§11. Invariants of Smooth Abelian Coverings

Let $\pi : X \to Y$ be an abelian cover with Galois group G, X smooth (so, it makes sense to work with K_X). Let $\{L_\chi\}_{\chi \in G^*}$ and $\{D_g\}_{g \in G}$ be the building data of π.

Theorem

$$h^i(\mathcal{O}_X) = h^i(\mathcal{O}_Y) + \sum_{\chi \neq 1} h^i(Y, L_\chi^{-1})$$

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) + \sum_{\chi \neq 1} \chi(L_\chi^{-1})$$

$$p_g(X) = p_g(Y) + \sum_{\chi \neq 1} h^0(Y, K_X + L_\chi)$$

proof $\pi_* \mathcal{O}_X \cong \bigoplus_{\chi \in G^*} L_\chi^{-1} \Rightarrow$

$$h^i(\mathcal{O}_X) = h^i(Y, \pi_* \mathcal{O}_X) = \sum_{\chi \in G^*} h^i(Y, L_\chi^{-1})$$

Instead, $p_g(X) = h^0(K_X) \underset{\text{Serre Duality}}{=} h^2(\mathcal{O}_X) = \sum_{\chi \in G^*} h^2(Y, L_\chi^{-1})$

$$= \sum_{\chi \in G^*} h^0(Y, K_Y + L_\chi).$$

(this holds only for surfaces; we present below a proof for the higher dimensional case)

Finally, $\chi(\mathcal{O}_X) = \sum_{i=0}^{\dim X} (-1)^i h^i(\mathcal{O}_X) = \sum_{\chi \in G^*} \left(\sum_{i=1}^{\dim X} (-1)^i h^i(Y, L_\chi^{-1}) \right)$

$$= \sum_{\chi \in G^*} \chi(L_\chi^{-1}) \qquad \blacksquare$$

Corollary For surfaces, we have

$$q(X) = q(Y) + \sum_{\chi \neq 1} h^1(L_\chi^{-1}),$$

$$\chi(\mathcal{O}_X) = |G|\chi(\mathcal{O}_Y) + \frac{1}{2}\sum_{\chi \neq 1} L_\chi(L_\chi + K_Y)$$

$$p_g(X) = p_g(Y) + \sum_{\chi \neq 1} h^0(Y, L_\chi + K_Y)$$

proof From Riemann-Roch Theorem for surfaces, we have that

$$\chi(D) = \chi(\mathcal{O}_X) + \frac{1}{2} D(D - K_X)$$

Thus,

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) + \sum_{\chi \neq 1} \underbrace{\chi(L_\chi^{-1})}$$

$$\chi(\mathcal{O}_Y) + \frac{1}{2}(-L_\chi)(-L_\chi - K_X)$$

$$\chi(\mathcal{O}_Y) + \frac{1}{2} L_\chi(L_\chi + K_X)$$

$$= |G|\chi(\mathcal{O}_Y) + \frac{1}{2}\sum_{\chi \neq 1} L_\chi(L_\chi + K_X) \qquad \blacksquare$$

Let us compute also the Topological Euler characteristic $e(X)$ when X is smooth.

Prop Let $\pi : X \to Y$ be a smooth ab. cover of surfaces with Galois Group G, and building data $\{L_\chi\}_{\chi \in G^*}, \{D_g\}_{g \in G}$. Then

$$\boxed{e(X) = |G|\left(e(Y) - \sum_{g \in G}\left(1 - \frac{1}{|g|}\right)e(D_g) + \frac{1}{2}\sum_{g \neq h}\left(1 - \frac{1}{|g|}\right)\left(1 - \frac{1}{|h|}\right)D_g \cdot D_h\right)}$$

Proof $e(X) = n_0 - n_1 + n_2 - n_3 + n_4$, where n_i is the number of cells of dimension i, and $e(X)$ is computed using a cellular decomposition of X.

We can choose a particular cellular decomp. of Y such that:

- the intersection points between D_g and D_h are 0-cells of the decomposit;
- The cellular decomp. of Y induces a cellular dec. on each D_g;
- The cells are contained on some D_g or they do not touch the branch locus D.

This particular decomposition induces a cellular decomps. on X.

If π is not ramified, then any cell of Y is counted $|G|$-times on X, so $e(X) = |G| e(Y)$.

Instead, if π is ramified, then we can add a correction term.

Indeed, any 2-cell is counted $|G|$-times with the exception of the components D_g, that are counted $\frac{|G|}{|g|}$-times.

Hence to $|G| e(Y)$ we have to subtract

$$\sum_{g\in G}{}' \left(|G| - \frac{|G|}{|g|}\right) e(D_g).$$

However, we subtracted too much. Indeed, the common points lying over some D_g and D_h are counted $\frac{|G|}{|\langle g,h\rangle|}$-times, namely $\frac{|G|}{|g|\cdot|h|}$-times since X is smooth and $\langle g\rangle \oplus \langle h\rangle \to G$ is injective.

However, we actually counted those 0-cells

$$\left[|G| - \left(|G| - \frac{|G|}{|g|}\right) - \left(|G| - \frac{|G|}{|h|}\right)\right]\text{-times}$$

so we need to add the correction term:

$$\frac{1}{2}\sum_{g\neq h}\left(\left(|G| - \frac{|G|}{|g|}\right) + \left(|G| - \frac{|G|}{|h|}\right) - \left(|G| - \frac{|G|}{|g||h|}\right)\right) D_g \cdot D_h$$

$\blacksquare$

We are interested to compute a (multiple) of a canonical divisor K_X in function of the building data of π.

Theorem Let $e :=$ the exponent of G, namely the higher order among the elements of G.

Then

$$\boxed{eK_X \equiv \pi^*\left(eK_Y + \sum_{g\in G} \frac{|g|-1}{|g|}\cdot e\, D_g\right)}$$

proof We apply Riemann-Hurwitz formula to π:

$$(*) \qquad K_X \equiv \pi^*(K_Y) + R_{\pi} = \pi^*(K_Y) + \sum_{g\in G}(|g|-1)R_g$$

the ramific. locus is made by the irred. compon. of R_g, $g\in G$. Eeach component of R_g has ramif. index $|g|$, so R_g appears in R_{π} with multiplicity $|g|-1$.

We proved in the previous lectures that $\pi^*(D_g) = |g|R_g$

So $\quad eR_g = \frac{e}{|g|}|g|R_g = \frac{e}{|g|}\pi^*(D_g) = \pi^*\left(\frac{e}{|g|}D_g\right) \Rightarrow$ by $(*)$

$$(*) \qquad eK_X \equiv \pi^*(eK_Y) + \sum_{g\in G}(|g|-1)eR_g =$$

$$\equiv \pi^*\left(eK_Y + \sum_{g\in G}\frac{|g|-1}{|g|}\cdot e\cdot D_g\right) \qquad \blacksquare$$

Corollary: $K_X^n = |G|\cdot\left(K_Y + \sum_{g\in G}\frac{|g|-1}{|g|}\cdot D_g\right)^n$

We want also to study the linear system $|K_X|$, which gives information of the canonical map $\varphi_{K_X}: X \dashrightarrow \mathbb{P}^{p_g-1}$, such as its base points.

The action of G on X induces a representation on $\pi_* \omega_X$, which then splits as a direct sum of eigensheaves of character $\chi \in G^*$:

$$\pi_* \omega_X = \bigoplus_{\chi \in G^*} (\pi_* \omega_X)^\chi$$

(The action of $g \in G$ on a local n-form $\omega = f dx_1 \wedge \cdots \wedge dx_n$ is $g \cdot \omega := (g)^* \omega$, namely $g \cdot (f dx_1 \wedge \cdots \wedge dx_n) = (f \circ g^{-1}) \, d(x_1 \circ g^{-1}) \cdots d(x_n \circ g^{-1})$).

We are interested to determine $(\pi_* \omega_X)^\chi$.

Theorem (Liedtke Formula)

$$\boxed{\pi_* \omega_X^\chi \cong \omega_Y \otimes \mathcal{L}_\chi}$$

The canonical system of X hase the following decomposition:

$$\boxed{H^0(X, K_X) = \bigoplus_{\chi \in G^*} \Big[\prod_{g \in G} t_g^{|g| - r_\chi^g - 1} \Big] \cdot \pi^* H^0(Y, K_Y + \mathcal{L}_\chi),}$$

where $(t_g = 0)$ is the local zero laws of R_g.

Remark $|K_X|$ is generated by all effective divisors $\pi^* |K_Y + \mathcal{L}_\chi| + \sum_{g \in G} (|g| - 1 - r_\chi^g) R_g$ such that $h^0(Y, K_Y + \mathcal{L}_\chi) \neq 0$.

proof Locally around $\mathrm{supp}(D_g)$, we have

$$(t,\dots x_n) \xrightarrow{g} (\xi t, x_2, \dots x_n)$$

and

$$(t,\dots x_n) \xrightarrow{\pi} (\underset{(z_1,}{t^{|g|}}, \underset{z_2,}{x_2}, \dots \underset{z_n)}{x_n}) \quad , \quad R_g = (t=0)$$

We observe that $(\pi_* \omega_X)^{\chi}$ is generated by the local parameter $t^{|g|-r_\chi^g-1} \cdot dt \wedge dx_2 \wedge \dots \wedge dx_n$. Indeed, given $\omega = f(t,\dots x_n) dx_1 \wedge \dots \wedge dx_n$, then

$$g\cdot \omega = \chi(g)\omega = g^* f \, d(g^* t) \wedge \dots \wedge d(g^* x_n) = f(\xi^{|g|-1} t, \dots x_n) \xi^{|g|-1} dt \wedge \dots \wedge dx_n$$

$$\xi^{r_\chi^g} f(t,\dots x_n) dt \wedge \dots \wedge dx_n$$

$$\Leftrightarrow \quad \xi^{r_\chi^g} f(t,\dots x_n) = f(\xi^{|g|-1} t, \dots x_n)\xi^{|g|-1} \Leftrightarrow$$

$$f(t,\dots,x_n) = \xi^{|g|-r_\chi^g-1} f(\xi^{|g|-1} t, \dots x_n)$$

$$f = \sum a_k(x_2,\dots x_n) t^k = \xi^{|g|-r_\chi^g-1} \cdot \sum a_k(x_2,\dots x_n) \underbrace{\xi^{(|g|-1)k}}_{= \xi^{-k}} t^k$$

$$= \sum a_k(x_2,\dots x_n) \xi^{|g|-r_\chi^g-1-k} t^k$$

So $a_k(x_2,\dots x_n) \neq 0 \Leftrightarrow \xi^{|g|-r_\chi^g-1-k} = 1 \Leftrightarrow |g|-r_\chi^g-1-k \equiv 0 \mod |g|$

$$\Leftrightarrow \quad k \equiv |g| - r_\chi^g - 1 \mod |g|$$

Thus,

$$f = \sum a_k(x_2,\dots x_n) (t^{|g|})^{\alpha_k} \cdot t^{|g|-r_\chi^g-1}$$

$$= t^{|g|-r_\chi^g-1} \cdot \left(\sum a_k(z_2,\dots z_n) z_1^{\alpha_k}\right)$$

$\Rightarrow$ a local parameter is $t^{|g|-r_\chi^g-1} \cdot dt \wedge dx_2 \wedge \dots \wedge dx_n$.

However, a local parameter for $\mathcal{L}_\chi^{-1}$ is $t^{r_\chi^g}$, so we have an isomorphism

$$\begin{array}{ccccc} (\pi_* \omega_X)^\chi \otimes \mathcal{L}_\chi^{-1} & \xrightarrow{\sim} & (\pi_* \omega_X)^1 & \xleftarrow[\text{pullback map}]{\sim} & \omega_Y \\ t^{|g|-r_\chi^g-1} dt \wedge \dots \wedge dx_n \otimes t^{r_\chi^g} & & \pi^*(dz_1 \wedge \dots \wedge dz_n) & \longleftarrow & dz_1 \wedge \dots \wedge dz_n \\ & \searrow & \parallel & & \\ & & |g|\, t^{|g|-1} dt \wedge dx_2 \wedge \dots \wedge dx_n & & \end{array}$$

Thus, $(\pi_* \omega_X)^\chi \otimes \mathcal{L}_\chi^{-1} \cong \omega_Y \implies (\pi_* \omega_X)^\chi \cong \omega_Y \otimes \mathcal{L}_\chi$.

Finally, $\pi^* H^0(X, \omega_Y \otimes \mathcal{L}_\chi) \hookrightarrow H^0(X, \omega_X)$

$$\pi^*(s) \longmapsto \left(\prod_{g \in G} t_g^{|g|-r_\chi^g-1} \right) \pi^*(s)$$

is injective. Indeed, given $\pi^* s \in \pi^* H^0(Y, \omega_Y \otimes \mathcal{L}_\chi)$,

Then $\operatorname{div}\left(\prod t_g^{|g|-r_\chi^g-1} \cdot \pi^* s\right) = \sum_{g \in G} (|g| - r_\chi^g - 1) R_g \cdot \operatorname{div}(\pi^* s)$

$$\equiv \sum_{g \in G} (|g| - r_\chi^g - 1) R_g + \pi^* K_Y + \underbrace{\pi^* \mathcal{L}_\chi}_{= \sum_{g \in G} r_\chi^g R_g} \equiv$$

$$\equiv \pi^* K_Y + \sum_{g \in G} (|g| - 1) R_g \underset{R-H}{\equiv} K_X \qquad \blacksquare$$

One can also investigate $(\pi_* \omega_X^{\otimes m})^\chi$ with $m \geq 1$. The case $m = 2$ has been made by Bauer-Pignatelli, while the general case can be found in the Notes of this course. Finally, one can also generalize the above formula and obtain a decomposition of

$$H^0(X, mK_X), \quad m \geq 1.$$

(This has been made by Alessandro, Frapporti, Gleissner, 2025).

Remark These formulas work also for X normal and not only smooth, but we need to give a precise meaning of K_X, which is not well defined anymore (and one could give different reasonable definitions of K_X, which are in general not equivalent, but still the same if X is smooth).

Thm

$$\left(\pi_* \omega_X^{\otimes m}\right)^X \cong \omega_Y^{\otimes m} \otimes \mathcal{L}_{X^{-1}} \otimes \mathcal{O}_Y\left(\sum_{g \in G} \left\lfloor \frac{(|g|-1)(m-1) + k_X^g}{|g|} \right\rfloor D_g\right)$$

where $k_X^g := \begin{cases} |g|-1 & \text{if } r_X^g = 0 \\ r_X^g - 1 & \text{if } r_X^g \neq 0 \end{cases}$.

$$\boxed{H^0(X, mK_X) = \bigoplus_{\chi \in G^*} \prod_{g \in G} t_g^{r_\chi^g - m + \left\lceil \frac{m - r_\chi^g}{|g|} \right\rceil \cdot |g|} \pi^* H^0\left(Y, \mathcal{O}_Y\left(\sum_{g \in G}\left(m - \left\lceil \frac{m - r_\chi^g}{|g|} \right\rceil\right) D_g\right) \otimes \mathcal{L}_\chi^{-1} \otimes \omega_Y^{\otimes m}\right)}$$

One can use these formula to compute the Kodaira dimension of X:

Thm (F.-Ulivi) X smooth.

Let $D := |G| K_Y + \sum_{g \in G} \frac{|G|(|g|-1)}{|g|} D_g$. Then the Kodaira dimension of X is the Iitaka dim. of D:

$$\boxed{\kappa(X) = \kappa(Y, D)}.$$

Example: Let us determine in general the Kodaira dim. of a smooth ab. cover of $\mathbb{P}^2$.

Firstly, we remind that given $D \equiv d \cdot H$, then

$$h^0(\mathbb{P}^2, D) = \begin{cases} \binom{d+2}{2} = \dfrac{(d+2)(d+1)}{2} & \text{if } d \geq 0 \\ 0 & \text{if } d < 0 \end{cases}.$$

\# lin. indip. homog. poly. in 3 variables.

Thus, the Iitaka dimension of a divisor D of $\mathbb{P}^2$ is

$$k(\mathbb{P}^2, D) = \begin{cases} -\infty & \text{if } d < 0 \\ 0 & \text{if } d = 0 \\ 2 & \text{if } d \geq 0 \end{cases}$$

Given a smooth $\pi : X \to \mathbb{P}^2$ with ab. group G, then

$$k(X) = k\Big(\mathbb{P}^2, |G| K_{\mathbb{P}^2} + \sum_{g \in G} \frac{|G|(|g|-1)}{|g|} D_g\Big)$$

$$= \begin{cases} -\infty & \text{if } \sum_{g \in G} \frac{|g|-1}{|g|} d_g < 3 \\ 0 & \text{if } \sum_{g \in G} \frac{|g|-1}{|g|} d_g = 3 \\ 2 & \text{if } \sum_{g \in G} \frac{|g|-1}{|g|} d_g \geq 3 \end{cases}$$

Exercise: Compute the Kodaira dimension of a smooth ab. cover of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_n$.